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LETTER TO THE EDITOR

Walks with ghosts

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Abstract. The supersymmetric formulation of the self-suppressing random walk is examined. By defining the theory as a lattice model it is shown that the supersymmetry remains unbroken throughout the physical parameter space, contrary to claims in the literature.

The self-avoiding walk is an important problem both in its own right and as a model for polymer physics. Analytic progress has been facilitated by the observation due to de Gennes [1] (see also Emery [2]) that the excluded volume problem is equivalent to an n-component spin system in the limit $n \to 0$. Subsequently, it was noted by McKane [3] and Parisi and Sourlas [4] that the limit may be avoided through introducing anticommuting ghost fields. Essentially the same model has been examined by Fujikawa [5] and its Becchi–Rouet–Stora (BRS) [6] supersymmetry exhibited: in d Euclidean dimensions the action is

$$S = \int \mathrm{d}^d x \, \mathrm{d}\theta [\phi(-\Delta + m^2)\psi + \tfrac{1}{2}\lambda\phi\psi\partial_\theta(\phi\psi)] \tag{1}$$

where

$$\begin{aligned} \phi(x,\theta) &= \overline{\Phi}(x) + \theta\bar{\xi}(x) \\ \psi(x,\theta) &= \xi(x) + \theta\Phi(x) \end{aligned} \tag{2}$$

are BRS superfields, θ, $\xi(x)$ and $\bar{\xi}(x)$ being elements of a Grassmann algebra [7]. Supersymmetry transformations correspond to translations in the coordinate θ

$$\begin{aligned} \mathrm{e}^{\lambda Q}\phi(x,\theta)\,\mathrm{e}^{-\lambda Q} &= \phi(x,\theta+\lambda) \\ \mathrm{e}^{\lambda Q}\psi(x,\theta)\,\mathrm{e}^{-\lambda Q} &= \psi(x,\theta+\lambda) \end{aligned} \tag{3}$$

generated by

$$Q = \int \mathrm{d}^d x (\bar{\xi}\partial/\partial\overline{\Phi} + \Phi\partial/\partial\xi) \tag{4}$$

and BRS invariance in (1) is a simple consequence of translational invariance.

It has been suggested, based on the saddle-point approximation, that for $m^2 < 0$ the model (1) exhibits spontaneous breaking of the BRS supersymmetry, accompanied by Goldstone bosons and fermions [4, 5], and that these modes are associated with a phase where the walks fill the space [4]. Of course, since the connection with the polymer problem is through a Laplace transform (m^2 being conjugate to the Gaussian surface), $m^2 \geqslant 0$ in that case. Furthermore, from general considerations on infrared divergences [8] such expectations cannot be fulfilled for $d = 2$; indeed, for $d = 2$ and $m^2 = 0$ Ziegler [9] has demonstrated the existence of a finite correlation length.

To address this issue more generally, consider regularizing the supersymmetric model by defining it on a lattice. Since the fermions here have spin-0 there is no problem in principle of doing so as long as all fields are subject to periodic boundary conditions, or else supersymmetry is broken explicitly. The partition function is

$$Z = \int \Pi_j \frac{\mathrm{d}\phi_j\,\mathrm{d}\psi_j}{\pi}\,\mathrm{d}\sigma_j\,\hat{f}(\sigma_j)\exp\left[-\mathrm{i}\int\,\mathrm{d}\theta\,\sigma_j\phi_j\psi_j\right]\exp\left[-\beta H\right] \tag{5}$$

where

$$H = -\int\,\mathrm{d}\theta\sum_{ij}K_{ij}\phi_i\psi_j \tag{6}$$

is the Hamiltonian ($K_{ij} = 1$ if i and j are nearest neighbours, $K_{ij} = 0$ otherwise), $\hat{f}(\sigma)$ is the fourier transform of

$$f(x) = \mathrm{e}^{-rx-gx^2/2} \tag{7}$$

and integration over superfields is defined as integration over components [10]

$$\mathrm{d}\phi\,\mathrm{d}\psi = \mathrm{dRe}\Phi\,\mathrm{dIm}\Phi\,\mathrm{d}\bar{\xi}\,\mathrm{d}\xi. \tag{8}$$

The parameters appearing are connected to those in the continuum theory (1) via

$$m^2 = (r\beta^{-1} - 2d)a^{-2} \qquad \lambda = g\beta^{-2}a^{d-4} \tag{9}$$

with a the lattice spacing. According to the functional formulation [11] of Witten's criterion [12], supersymmetry breaking requires $Z = 0$ whereas in the unbroken case $Z = 1$. Here we exploit the random walk representation [13, 14], and in particular the fact that for $g > 0$, $\lim_{x\to\infty} f(x)\,\mathrm{e}^{cx} = 0$ for all c, so that the σ integration contour can be deformed to provide $\mathrm{Im}\,\sigma$ sufficiently large and negative to ensure the existence of the Gaussian integral in (5) irrespective of $\beta \geqslant 0$. Hence, for all $g > 0$, $\beta \geqslant 0$ and r arbitrary

$$Z = \int \Pi_i\,\mathrm{d}\sigma_i\,\hat{f}(\sigma_i)\frac{\det[\mathrm{i}\sigma - \beta K]}{\det[\mathrm{i}\sigma - \beta K]} = \int \pi_i\,\mathrm{d}\rho_i\,f(\rho_i)\delta(\rho_i) = 1 \tag{10}$$

i.e. the supersymmetry is unbroken.

Defining correlators of the supersymmetric model as

$$\langle O\rangle = \int \Pi_j \frac{\mathrm{d}\phi_j\,\mathrm{d}\psi_j}{\pi}\,\mathrm{d}\sigma_j\,\hat{f}(\sigma_j)\exp\left[-\mathrm{i}\int\,\mathrm{d}\theta\,\sigma_j\phi_i\psi_j\right]\exp\left[-\beta H\right] \tag{11}$$

one has by supertranslational invariance

$$\langle\phi_i(\theta)\psi_j(\theta')\rangle = (\theta' - \theta)G_{ij}. \tag{12}$$

Indeed, using the properties of (Grassmann) Gaussian integrals, (12) is verified; in component form

$$\begin{aligned} G_{ij} &= \langle\bar{\Phi}_i\Phi_j\rangle = -\langle\bar{\xi}_i\xi_j\rangle \\ &= \sum_{wij}\beta^{|w|}\Pi_k P(n_k(w)) \end{aligned} \tag{13}$$

where the sum extends over all random walks from site i to site j of length $|w|$ and $n_k(w)$ is the number of times the kth site is visited. The $P(n)$ are given by

$$P(n) = [(n-1)!]^{-1}\int_0^\infty\,\mathrm{d}t\,t^{n-1}f(t) \qquad n \geqslant 1 \tag{14}$$

and, with $P(0) \equiv 1$, they obey the recurrence relation

$$g(n-1)P(n) = P(n-2) - rP(n-1) \qquad n \geqslant 2. \tag{15}$$

Note that the redundant parameter r may be fixed by the condition $P(1) = 1$. Strictly, for finite g, (13) and (14) describe the self-suppressing random walk. The self-avoiding walk may be obtained by the limit $g \to \infty$ with

$$r(g \to \infty) \to -\sqrt{g \ln\left(\frac{g}{2\pi}\right)} \tag{16}$$

such that $P(n > 1) = 0$. We observe that (16) is different from the usual nonlinear limit $r(g \to \infty) \to -g$—indeed if we try to construct a naive nonlinear model by taking $\hat{f}(\sigma_j) = \mathrm{e}^{\mathrm{i}\sigma_j}$ in (5) then $Z = 0$ and supersymmetry is broken, albeit we lose the connection to the random walk.

For finite g our results (12) and (13) do not preclude a massless phase for $d > 2$, but do imply that all the modes should be massless to respect the supersymmetry. In this regard we note that the mean-field approximation of [4] implies again $Z = 1$ and the Ward identity $\langle\bar{\Phi}\Phi\rangle = -\langle\bar{\xi}\xi\rangle$.

Hence, if bosons condense, $\langle\bar{\Phi}\Phi\rangle \neq 0$, then ghosts also condense, $\langle\bar{\xi}\xi\rangle \neq 0$, and taking into account both condensates the mass of the nominal Higg's mode in fact vanishes. It would certainly be of interest to investigate the phase structure of the supersymmetric model by numerical simulation; however, this lies beyond the scope of the present work.

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